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A decomposition of the adjoint representation of $U_q(sl_2)$

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Quantum algebra. First, we introduce notation.

DEFINITION. *Let $U_q^{(l)}$ be an associative algebra / $K = \mathbb{Q}(q)$ (q is an indeterminate.), defined by a system of generators; $e, f, k^{\frac{1}{2}}, k^{-\frac{1}{2}}$, and their relations:*

$$k^{\frac{1}{2}}k^{-\frac{1}{2}} = 1, \quad k^{-\frac{1}{2}}k^{\frac{1}{2}} = 1$$
$$k^{\frac{1}{2}}ek^{-\frac{1}{2}} = qe \quad k^{\frac{1}{2}}fk^{-\frac{1}{2}} = q^{-1}f$$
$$ef - fe = \frac{k^2 - k^{-2}}{q^2 - q^{-2}}$$

As usual, we give U_q a Hopf algebra structure by equipping it with

$$\begin{aligned}\Delta : \quad & e \longmapsto e \otimes k^{-1} + k \otimes e \\ & f \longmapsto f \otimes k^{-1} + k \otimes f \\ & k^{\frac{1}{2}} \longmapsto k^{\frac{1}{2}} \otimes k^{\frac{1}{2}}\end{aligned}$$

$$S : \quad e \mapsto -q^{-2}e, \; f \mapsto -q^2 f, \; k^{\frac{1}{2}} \mapsto k^{-\frac{1}{2}}$$
$$\varepsilon : \quad e \mapsto 0, \; f \mapsto 0, \; k^{\frac{1}{2}} \mapsto 1$$

DEFINITION.
(1) *$U_q^{(m)}$ denotes the subalgebra of $U_q^{(l)}$ generated by e, f, k, k^{-1}.*
(2) *$U_q^{(s)}$ denotes the subalgebra of $U_q^{(m)}$ generated by $E = ek$, $F = k^{-1}f$, $K = k^2$, K^{-1}.*

REMARK. *If we choose another system of generators; E, F, $k^{\frac{1}{2}}$, $k^{-\frac{1}{2}}$ for $U_q^{(l)}$, then Δ and S become*

$$\begin{aligned}\Delta : \quad & E \longmapsto E \otimes 1 + k^2 \otimes E \\ & F \longmapsto F \otimes k^{-2} + 1 \otimes F\end{aligned}$$

$$S : \quad E \mapsto -k^{-2}E \; , \; F \mapsto -Fk^2$$

We put, $C = fe + \frac{q^2k^2 + q^{-2}k^{-2}}{(q^2 - q^{-2})^2}$

Adjoint Representation.

DEFINITION. *$U_q^{(l)}$ becomes a $U_q^{(l)}$-module by*

$$
\begin{aligned}
Ad(e)x &= exk - q^{-2}kxe \\
Ad(f)x &= fxk - q^{2}kxf \qquad (\ x \in U_q^{(l)} \ ) \\
Ad(k^{\frac{1}{2}})x &= k^{\frac{1}{2}}xk^{-\frac{1}{2}}
\end{aligned}
$$

We denote it by (Ad, U_q^{ad}), and we call it the adjoint representation.

DEFINITION. *We define submodules of U_q^{ad} as follows:*

$$
\begin{aligned}
V_{\alpha+\frac{1}{2}} &= Ad(U_q^{(l)})k^{\alpha+\frac{1}{2}} && (\alpha \in \mathbb{Z}) \\
V_{2\alpha+1} &= Ad(U_q^{(l)})k^{2\alpha+1} && (\alpha \in \mathbb{Z}) \\
V_{2\alpha} &= Ad(U_q^{(l)})C^{-\alpha}k^{2} && (\alpha \in \mathbb{Z}_{\leqq 0}) \\
V_{2\alpha} &= Ad(U_q^{(l)})k^{-\alpha+2}e^{\alpha} + Ad(U_q^{(l)})k^{-\alpha+2}f^{\alpha} && (\alpha \in \mathbb{Z}_{>0}) \\
V_{loc} &= \oplus_{n\geq 0}K[C]Ad(U_q^{(l)})k^{-n}e^{n}
\end{aligned}
$$

DEFINITION.

(1)

$$X(d) = U_q^{(l)}/U_q^{(l)}(k^{\frac{1}{2}} - q^{d}) \quad (d \in \mathbb{Z})$$

It has a naturally induced $U_q^{(l)}$-module structure using the left regular representation of $U_q^{(l)}$.

We put,

$$v_d = 1 \ mod \ U_q^{(l)}(k^{\frac{1}{2}} - q^{d})$$

(2) *$L(d)$ is the irreducible module with the highest weight q^{2d}.*

(3) *τ is a K-algebra automorphism which sends e, f, $k^{\frac{1}{2}}$ to f, e, $k^{-\frac{1}{2}}$ respectively.*

REMARK. *τ induces an isomorphism between $X(d)$ and $X(-d)$.*

Decomposition of the Adjoint Representation.

THEOREM.

(1)

$$U_q^{ad} = V_{loc} \oplus (\oplus_{n\in\frac{1}{2}\mathbb{Z}} V_n)$$

(2)

$$U_q^{(m)} = V_{loc} \oplus (\oplus_{n\in\mathbb{Z}} V_n)$$

(3)
$$U_q^{(s)} = V_{soc} \oplus (\oplus_{n \in 2\mathbb{Z}} V_n)$$

(4) *Any irreducible submodule of* U_q^{ad} *is contained in* V_{soc}.
(5) V_n ($n \in \frac{1}{2}\mathbb{Z}$)*'s are indecomposable modules.*
(6) $V_{2\alpha}$ ($\alpha \in \mathbb{Z}_{>0}$) *is isomorphic to*

$$X(\alpha) \oplus X(-\alpha)/(-id \oplus \tau)(U_q^{(l)} f^{\alpha} v_{\alpha})$$

V_β ($\beta \notin 2\mathbb{Z}_{>0}$)*'s are isomorphic to* $X(0)$.
(7) $X(0)$ *and* $V_{2\alpha}$*'s* ($\alpha \in \mathbb{Z}_{>0}$) *are mutually non-isomorphic.*
(8) *If a direct summand of* U_q^{ad} *is finitely generated and is indecomposable, then it is isomorphic to a* $L(d)$, *or a direct summand of* $X(0)^{\oplus s} \oplus (\oplus_{j=1}^{r} V_{2j})$.

References

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